Exact Exponential Algorithms

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School on Algorithms, July 2021, IIIT Bangalore

NP-Complete Problems

NP-Complete Problems

Max. Independent Set

MaxCut

Vertex Cover

Minimum Dominating Set

3-SAT

3- Colouring

Travelling Salesman

Minimum Set Cover

Integer Linear Programming

Minimum Feedback Vertex Set

Minimum Guarding

Steiner Tree

Every problem can be solved by exhaustive search of the solution space !!!

Can we do better than exhaustive search?

- Assume exhaustive search takes $O(n^n)$ time ....

- Can we find an algorithm that runs in $O(2^n)$ time ?

- In this talk, we will see how to design good exponential algorithms

Exact Exponential Algorithms

by Fedor V. Fomin , Dieter Kratsch

Branching

Maximum Independent Set

- Graph G(V, E)
- Independent Set - A subset of vertices $V_1 \subset V$ such that any two vertices in V_1 are not adjacent
- Find an independent set of maximum cardinality.

Maximum Independent Set

- For any $v \in V$,
  - Maximum Independent set contains v

    OR

  - Maximum Independent set does not contain v

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Maximum Independent Set

- Branch on
  - select v - delete all the neighbours of v
  - discard v - delete v

- Recursively solve the two smaller subproblems

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```
MIS(G = (V, E))

{
      Let v be a vertex in V
      return max(1 + MIS(G − N[v]), MIS(G − v));
}
```

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Did we improve the running time?

$$T(n) = T(n-1) + T(n - d(v) - 1)$$

Did we improve the running time?

$$T(n) = T(n-1) + T(n-4)$$

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Did we improve the running time?

$$T(n) = T(n-1) + T(n-4)$$

$$T(n) = \mathcal{O}(1.38^n)$$

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3-SAT

- V : n variables - $x_1, x_2, \ldots\ldots, x_n$
- A CNF formula Φ with m clauses - where every clause has at most 3 literals

$$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_4 \vee \overline{x_1} \vee \overline{x_5}) \wedge (x_3 \vee \overline{x_2} \vee x_5) \wedge \ldots\ldots.$$

- Does there exist an assignment of V that satisfies Φ?

- For a clause, there is one assignment of 3 variables that does not satisfy the clause

- There are 7 assignments that can satisfy the clause

- Branch on these 7 assignments

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$$T(n) = 7T(n-3)$$

$$T(n) = \mathcal{O}(1.9^n)$$

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- For a clause $l_1 \vee l_2 \vee l_3$, branch on the following assignments
  - $l_1 = T$
  - $l_1 = F, l_2 = T$
  - $l_1 = F, l_2 = F, l_3 = T$

$$T(n) = T(n-1) + T(n-2) + T(n-3)$$

$$T(n) = \mathcal{O}(1.8^n)$$

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Dynamic Programming

Graph Coloring

- Given graph G(V, E)
- Assign colours to the vertices such that end points of every edge is assigned different colours
- Minimise the number of colours used

Graph Coloring

Graph Coloring

For $X \subseteq V$, let

OPT[X] = minimum number of colours required to proper colour G[X]

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Graph Coloring

For $X \subseteq V$, let

OPT[X] = minimum number of colours required to proper colour G[X]

$$\text{OPT}[X] = \min_{\substack{I \subseteq X \\ \text{I is independent}}} \{ 1 + \text{OPT}[X \setminus I] \}$$

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Graph Coloring

```
for i from 3 to n
  for all X ⊆ V of size i
    for all I ⊆ X which is an independent set
      OPT[X] =min { OPT [X], { 1 + OPT[X \ I] }}
return OPT[V]
```

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Graph Coloring

A graph with k vertices has at most $3^{\frac{k}{3}}$ maximal independent sets

$$T(n) = \mathcal{O}(2.44^n)$$

Travelling Salesman Problem

- Given n cities
- distances between each pair of cities
- What is the shortest possible route that visits every city exactly once and return to the origin city?

Travelling Salesman Problem

- Given n cities $\{c_1, c_2, \ldots\ldots, c_n\}$
- distances between each pair of cities $d(c_i, c_j)$
- What is the shortest possible route that visits every city exactly once and return to the origin city?

Travelling Salesman Problem

- OPT[S, c_i] = minimum length of a tour which starts in c_1, visits all cities from S and ends in c_i

Travelling Salesman Problem

$$\mathrm{OPT}[S, c_i] = \min_{c_j \in S \setminus \{c_i\}} \{ \, d(c_j, c_i) + \mathrm{OPT}[S \setminus \{c_j\}, c_j] \, \}$$

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Travelling Salesman Problem

$$\mathrm{OPT}[S, c_i] = \min_{c_j \in S \setminus \{c_i\}} \{ d(c_j, c_i) + \mathrm{OPT}[S \setminus \{c_j\}, c_j] \}$$

Running Time - $\mathcal{O}(2^n)$

Minimum Set Cover Problem

- Universe of n elements, $U = \{a_1, a_2, \ldots, a_n\}$
- Family of m subsets of U, $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$
- Find a minimum cardinality subset of $\mathcal{F}$ that covers U

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Minimum Set Cover Problem

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Minimum Set Cover Problem

- For $X \subseteq U$,

OPT[X, i] = minimum cardinality of a subset of $\{F_i, F_{i+1}, \ldots\ldots, F_m\}$ that can cover X

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Minimum Set Cover Problem

- For $X \subseteq U$,

$$OPT[X, i] = \min\{1 + OPT[X \setminus F_i, i + 1], OPT[X, i + 1]\}$$

Algebraic Techniques

Inclusion- Exclusion Principle

- Let A and B be sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

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Inclusion- Exclusion Principle

- Let $A_1, A_2, \ldots\ldots, A_n$ be sets

$$|\bigcup_{i\in[n]} A_i| = \sum_{X\subset[n]} (-1)^{|X|+1} |\bigcap_{i\in X} A_i|$$

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Inclusion- Exclusion Principle

- Let $A_1, A_2, \ldots\ldots, A_n$ be sets

$$|\bigcap_{i\in[n]} A_i| = \sum_{X\subseteq[n]} (-1)^{|X|} |\bigcap_{i\in X} (U \setminus A_i)|$$

Graph Colouring

- Given graph G(V, E)
- Is G k-colourable?

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Graph Colouring

- Given graph G(V, E)
- V can be coloured using k colours $\Longrightarrow$ V can be covered using k independent sets

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Graph Colouring

- Given graph G(V, E)
- V can be coloured using k colours $\Longrightarrow$ V can be covered using k independent sets

- U - set of tuples of k independent sets $(I_1, I_2, \ldots, I_k)$

- $$A_v = \{(I_1, I_2, \ldots, I_k) \in U : v \in \bigcup_{j=1}^{k} I_j\}$$

  A_v - set of tuples that cover the vertex v

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Graph Colouring

- Given graph G(V, E)
- V can be coloured using k colours $\Longrightarrow$ V can be covered using k independent sets

$$\left|\bigcap_{v \in V} A_v\right|$$ - number of k - colourings of V

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Graph Colouring

- Given graph G(V, E)
- V can be coloured using k colours $\Longrightarrow$ V can be covered using k independent sets

$$|\bigcap_{v\in V} A_v|$$ - number of k - colourings of V

$$|\bigcap_{i\in[n]} A_i| = \sum_{X\subseteq[n]} (-1)^{|X|} |\bigcap_{i\in X} (U \backslash A_i)|$$

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Graph Colouring

- To compute $|U \backslash A_v|$

$|U \backslash A_v|$ - (#independent sets in $G \backslash \{v\})^k$

Graph Colouring

- Graph Coloring can be solved in $\mathcal{O}(2^n)$ time

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"..for every polynomial-time algorithm you have, there is an exponential algorithm that I would rather run"

Alan Perlis (Turing Award Winner)